Dispersion effects of extensional waves in pre-stressed imperfectly bonded incompressible elastic layered composites

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Abstract
The effect of an imperfect interface, on time-harmonic extensional wave propagation in a pre-stressed symmetric layered composite is considered. The bimaterial composite consists of incompressible isotropic elastic materials. The shear spring type resistance model employed to simulate the imperfect interface can accommodate the extreme cases of perfect bonding and a fully slipping interface. The dispersion relation obtained by formulating the incremental boundary-value problem and the use of the propagator matrix technique, is analyzed at the low and high wavenumber limits. For the perfectly bonded and imperfect interface cases in the low wavenumber region, only the fundamental mode has a finite phase speed, while other higher modes have an infinite phase speed when the dimensionless wavenumber approaches zero. However, for the fully slipping interface in the low wavenumber region, both the fundamental mode and the next lowest mode have finite phase speeds. In the high wavenumber region, when the dimensionless wavenumber tends to infinity, the phase speeds of the fundamental mode and the higher modes depend on the phase speeds of the surface and interfacial waves and on the limiting phase speed of the composite. An expression to determine the cut-off frequencies is obtained from the dispersion relation. Numerical examples of dispersion curves are presented, where when the material has to be prescribed either Mooney–Rivlin material or Varga material is assumed. The effect of the imperfect interface is clearly evident in the numerical results.

Keywords: Wave propagation; Pre-stressed layered composite; Dispersion curves; Imperfect interface

1. Introduction
The effect of pre-stress on waves propagating in an incompressible elastic layer has been studied by many researchers such as Ogden and Roxburgh [9], Rogerson and Fu [14] and Rogerson [13]. The former paper presents a good review of wave propagation and stability problems in pre-stressed media, while the two latter papers give more details about the dispersion relation and its asymptotic behavior. The problem of wave propagation in a pre-stressed layered half space was considered by Ogden and Sotiropoulos [10] while Sotiropoulos and Sifniotopoulou [20] considered wave propagation in a layer between two half spaces. Motivated by the wide usage of layered composites in engineering practice, wave propagation in pre-stressed layered composites has been considered by Rogerson and Sandiford [15–18].
In general, layered composites are assumed to be perfectly bonded, although in practice due to defects in manufacturing or due to loading, the interface may be unable to perfectly transfer stresses due to displacement discontinuities at the interface. In linear elastic wave propagation problems, the effects of an imperfect interface have been widely studied both analytically and experimentally. Generalized boundary conditions for an imperfect interface were presented by Rokhlin and Wang [19] while Martin [7] has reviewed imperfect interface models. Ultrasonic techniques combined with analytical theories are also used to investigate imperfect interfaces as reported by Pilarski and Rose [12] and Kundu and Maslov [6]. A recent review article by Chimenti [1] contains a vast number of references on the use of ultrasonic methods in the examination of layered composites.

In the present analysis, the effect of an imperfect interface on time-harmonic extensional wave propagation in a pre-stressed incompressible symmetric layered composite is studied. The linear shear spring type resistance model is employed to simulate the imperfect interface. The advantage of this resistance model is that both perfect bonding and a fully slipping interface can be accommodated by such a model. The basic theory and equations of infinitesimal harmonic wave propagation in pre-stressed incompressible elastic media are given in Section 2. In Section 3, the propagator matrix method in linear elastic wave and vibration problems first presented by Gilbert and Backus [4] and later adopted for pre-stressed layered composites by Rogerson and Sandiford [16] is used to derive the dispersion relation for harmonic extensional wave propagation. The effect of the imperfect interface on the dispersion relation at low and high wavenumber limits are studied, and the limiting phase speeds and the cut-off frequencies are discussed in Section 4. Asymptotic expansions of the dispersion relation are not included in this paper and will be reported elsewhere. In Section 5, four numerical examples of pre-stressed layered composites are considered, where when necessary Mooney–Rivlin and Varga materials are assumed and the phase speeds and frequencies of the fundamental mode and the next 15 modes are plotted for different values of the shear spring parameter.

2. Basic equations

In this section, the basic equations for infinitesimal time-harmonic wave propagation in pre-stressed incompressible elastic media are presented. Details of the derivation, can be found in [2,16]. Consider a homogeneous incompressible isotropic elastic material with an initial unstressed state denoted by \( B_u \), which after a homogeneous quasi-static large deformation has a new configuration denoted by \( B_e \), which is the pre-stressed equilibrium state. A Cartesian coordinate system \( O_{x_1x_2x_3} \), with axes coincident with the principal axes of the right Cauchy–Green strain tensor, is chosen for the configuration \( B_e \). Let \( u \) be a small, time dependent displacement superimposed on \( B_e \). The incremental equations of motion, where the linearized incompressibility condition \( \delta_i u_i = 0 \) has been used is

\[
A_{ijkl} u_{ij} - p,_i = \rho \ddot{u}_i,
\]

in which \( A_{ijkl} \) are the components of the fourth-order tensor of first-order instantaneous elastic moduli which relates the nominal stress increment tensor and the deformation gradient increment tensor and can be given in terms of derivatives of the strain energy function (p. 344 of [8]), \( p \) the incremental pressure, \( \rho \) the material density and superimposed dot and comma indicate differentiation with respect to time \( t \) and spatial coordinate component in \( B_e \), respectively.

For the plane strain incremental problem the incremental displacement components \( u_1 \) and \( u_2 \) are independent of \( x_3 \) and \( u_3 = 0 \), Eq. (2.1) then reduces to

\[
\begin{align*}
A_{1111} & \ddot{u}_{11} + (A_{1122} + A_{2211}) \ddot{u}_{22} + A_{1212} \ddot{u}_{12} - p,1 = \rho \ddot{u}_1, \\
(A_{1122} + A_{2211}) & \ddot{u}_{11} + A_{1212} \ddot{u}_{22} + A_{2222} \ddot{u}_{22} - p,2 = \rho \ddot{u}_2.
\end{align*}
\]

The components of the nominal stress increment tensor relative to the configuration \( B_e \) are expressed as

\[
\sigma_{ij}(x_1, x_2, t) = A_{ijkl} \ddot{u}_{ij} + \dot{p} \ddot{u}_j - \bar{p} \ddot{u}_i.
\]
and explicit expressions for \( s_{021} \) and \( s_{022} \) are written as

\[
s_{021}(x_1, x_2, t) = A_{02111}u_{1,1} + (A_{02112} + \tilde{p})u_{2,1}, \quad s_{022}(x_1, x_2, t) = A_{02211}u_{1,1} + (A_{02212} + \tilde{p})u_{2,2} - p.
\]

where \( \tilde{p} \) is a quasi-static pressure which is related to the principal Cauchy stress \( \sigma_2 \) in the \( x_2 \)-direction by \( \sigma_2 = A_{02112} - A_{02121} - \tilde{p} \).

For harmonic waves propagating in \( x_1 \)-direction, the solution of Eq. (2.2) may be expressed as

\[
(a_1, u_2, p) = (A_1, A_2, \lambda P) e^{ikx_1 e^{i\omega t}},
\]

where \( k \) is the wavenumber, \( \omega \) the phase speed, \( A_1, A_2 \) and \( P \) are unknown coefficients and the parameter \( q \) is to be determined. Substituting Eq. (2.5) into Eq. (2.2) and using the linearized incompressibility condition, yields a system of homogeneous equations for which a non-trivial solution exists provided that

\[
y^n + (\rho \omega^2 - 2\beta q^2 + (\alpha - \rho \omega^2)) = 0,
\]

where \( \alpha = A_{01112}, 2\beta = A_{01111} + A_{02222} - 2A_{01122} - 2A_{02122} \) and \( y = A_{00211} \). From the definition of instantaneous elastic moduli of incompressible isotropic elastic material, the parameters \( \alpha, \beta \) and \( y \) are expressed in terms of the strain energy function \( W \) and the principal stretches \( \lambda_1 \) and \( \lambda_2 \) as [3]

\[
a\lambda_1^2 = \frac{\lambda_1^2 W_{11} - \lambda_1^2 W_{22}}{\lambda_1^2 - \lambda_2^2}, \quad 2\beta + 2\gamma = \lambda_1^2 W_{11} + \lambda_2^2 W_{22} - 2\lambda_1 \lambda_2 W_{12} + 2\lambda_2 W_1;
\]

where \( W_{ij} = \partial W/\partial \epsilon_{ij} \), \( W_i = \partial W/\partial \epsilon_i \), \( (i, j = 1, 2) \) and when \( \lambda_1 = \lambda_2 \) Eq. (2.7) reduces to \( a = \beta = \gamma = (1/2)\lambda_1(W_{11} - \lambda_1 W_{12} + W_1) \).

In order to obtain the propagator matrix, the incremental displacements and stresses in Eqs. (2.4) and (2.5) are written in the form of a \( 4 \times 4 \) vector as

\[
(a_1, u_2, s_{021}, s_{022})^T = [U_{1}(x_2), U_{2}(x_2), S_{021}(x_2), S_{022}(x_2)]^T e^{ikx_1 e^{i\omega t}}.
\]

After some mathematical manipulation it can be shown that

\[
y(x_2) = HE(x_2)\mathbf{a},
\]

where \( y(x_2) \) is a displacement–stress increment vector and \( E(x_2) \) is a diagonal matrix given by

\[
y(x_2) = -[U_{1}(x_2), U_{2}(x_2), S_{021}(x_2), S_{022}(x_2)]^T, \quad E(x_2) = \text{diag}(e^{i\epsilon \omega x_2}, e^{-i\epsilon \omega x_2}, e^{i\epsilon \omega x_2}, e^{-i\epsilon \omega x_2}),
\]

and \( \mathbf{a} \) is a vector of arbitrary constants and \( H \) is a \( 4 \times 4 \) matrix which is independent of position \( x_2 \) defined by

\[
\mathbf{a} = (A_1^{(1)}, A_2^{(1)}, A_1^{(2)}, A_2^{(2)})^T, \quad H = \begin{bmatrix}
q_1 & -q_1 & q_2 & -q_2 \\
1 & 1 & 1 & 1 \\
\gamma f(q_1) & \gamma f(q_1) & \gamma f(q_2) & \gamma f(q_2) \\
\gamma g_1 f(q_2) & -\gamma g_1 f(q_2) & -\gamma g_2 f(q_1) & -\gamma g_2 f(q_1)
\end{bmatrix}.
\]

where \( f(q_m) = 1 + q_m^2 - \sigma \) and \( m = 1, 2 \) and \( \sigma = \sigma_2/y \).

The arbitrary constant vector \( \mathbf{a} \) is eliminated from Eq. (2.9) by introducing the vector \( y(x_2) \) at some location \( x_2 = \bar{x}_2 \) to yield

\[
y(x_2) = HE(x_2)E^{-1}(x_2)H^{-1}y(x_2) = P(x_2 - \bar{x}_2)\mathbf{y}(\bar{x}_2).
\]

The matrix \( P(x_2 - \bar{x}_2) \) is called the propagator matrix [4,16] and the components of \( P \) are given in Appendix A.
3. Formulation of the problem

Consider the pre-stressed symmetric layered composite shown in Fig. 1, which consists of two isotropic incompressible elastic materials and where the principal axes of the right Cauchy–Green strain tensor in each layer are coincident. The Cartesian coordinate system is chosen such that \(x_1\) and \(x_2\) axes are also coincident with the principal axes, the \(x_2\)-direction is normal to the free surface of the layered composite, wave propagation is in \(x_1\)-direction and the origin \(O\) lies at the mid plane of the composite. The thickness of the inner layer is \(2d\), and the thickness of the outer layers is \(h\). The outer layers and inner layer are homogeneous with material parameters and mass density \(\alpha, \beta, \gamma, \rho\) and \(\alpha^*, \beta^*, \gamma^*, \rho^*\), respectively. In the remainder of the paper, all quantities with an asterisk refer to variables and parameters of the inner layer.

Due to the symmetric geometry of the composite and the symmetric nature of extensional waves, only the upper half of the composite \((0 \leq x_2 \leq d + h)\) is considered. For the outer layer from Eq. (2.12) the relation between displacement–stress increment vectors at the boundary of the layered composite and the interface is written as

\[
y(d + h) = P(h)y(d).
\]

(3.1)

Replacing parameters \(q, \alpha, \beta, \gamma, \rho\) by \(q^*, \alpha^*, \beta^*, \gamma^*, \rho^*\) in Eq. (2.6), the displacement–stress increment vector and propagator matrix for the inner layer may be determined and are denoted by \(y^*\) and \(P^*\), respectively. For the inner layer the relation between displacement–stress increment vectors at the interface and mid-plane may be established as

\[
y^*(d) = P^*(d)y^*(0).
\]

(3.2)

The mid-plane conditions and the incremental traction free upper surface conditions can be written as

\[
U^*_{2}(0) = S^*_{21}(0) = 0, \quad S^*_{422}(d + h) = 0.
\]

(3.3)

At the interface, stress increments and the displacement increment in the \(x_2\)-direction are assumed to be continuous, while the shear stress increment is assumed to be proportional to the displacement increment jump in the \(x_1\)-direction. These interfacial conditions yield

\[
S^*_{021}(d) = S_{021}(d), \quad S^*_{022}(d) = S_{022}(d), \quad U^*_{2}(d) = U_{2}(d), \quad S^*_{422}(d) = \frac{k_x}{h}(U_{1}(d) - U^*_{1}(d)).
\]

(3.4)

where \(k_x\) is the non-dimensional shear spring parameter.

Using Eqs. (3.1) and (3.2) to satisfy the conditions in Eqs. (3.3) and (3.4) results in a system of four homogeneous equations in four unknowns, from which the dispersion relation for extensional waves in an imperfectly bonded composite is obtained as

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} \left[ P_{ij} P^*_{ij} - P_{ij} P^*_{ij} \right] + \frac{h b}{k_x} \sum_{i=1}^{4} \left[ P_{ij} P^*_{ij} - P_{ij} P^*_{ij} \right] - P_{ij} P^*_{ij} = 0.
\]

(3.5)

where \(P_{ij}\) and \(P^*_{ij}\) are the components of \(P(h)\) and \(P^*(d)\), respectively.

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Fig. 1. Pre-stressed equilibrium configuration of a symmetric layered composite.
The components of $P(b)$ and $P^*(d)$ from Appendix A are substituted into Eq. (3.5) and the common factor $q^2(q_1^2)^2 q_2^2 q_3^2 - q_1^2(q_1^2)^2 q_2^2 q_3^2$ is removed from the denominator. The removal of this common factor leads to spurious roots in the resulting relation

$$2q_1 f(q_1) f(q_2) f(q_3) f(q_4) + f(q_1) f(q_2) f(q_3) f(q_4) [C_1 S_1 + S_1] + f(q_1) f(q_2) f(q_3) f(q_4) [C_1 C_1] + S_1 S_2 D_4 + S_1 C_2 D_3]$$

$$- q_1 f(q_1)^2 [S_1 C_2 D_2 + S_1 C_2 D_3 + C_1 C_2 D_3 + C_1 C_2 D_4]$$

$$+ \left( \frac{k_b}{k_c} \right) q_1 f(q_1)^2 [C_1 S_1 D_4 + S_1 C_2 D_4 + (C_1 C_2 - 1) D_4]$$

$$- q_1 f(q_1)^2 [S_1 C_2 D_4 + (C_1 C_2 - 1) D_4 + S_1 S_2 D_4] = 0,$$

(3.6)

where

$$\Delta_1 = q_1 \left[ f^*(q_1) - f(q_1) \right] \left[ f(q_1) - f(q_2) \right] \left[ f(q_1) - f(q_3) \right] \left[ f(q_1) - f(q_4) \right] \left[ f(q_1) - (q_1)^2 S_1 C_2 \right] \left[ f(q_1) - (q_1)^2 S_1 C_2 \right],$$

$$\Delta_2 = q_2 \left[ f^*(q_2) - f(q_2) \right] \left[ f(q_2) - f(q_3) \right] \left[ f(q_2) - f(q_4) \right] \left[ f(q_2) - (q_2)^2 S_2 C_2 \right] \left[ f(q_2) - (q_2)^2 S_2 C_2 \right],$$

$$\Delta_3 = q_3 \left[ f^*(q_3) - f(q_3) \right] \left[ f(q_3) - f(q_4) \right] \left[ f(q_3) - (q_3)^2 S_3 C_2 \right] \left[ f(q_3) - (q_3)^2 S_3 C_2 \right],$$

$$\Delta_4 = q_4 \left[ f^*(q_4) - f(q_4) \right] \left[ f(q_4) - f(q_1) \right] \left[ f(q_4) - f(q_2) \right] \left[ f(q_4) - f(q_3) \right] \left[ f(q_4) - (q_4)^2 S_1 C_2 \right] \left[ f(q_4) - (q_4)^2 S_1 C_2 \right].$$

(3.7)

and $f(q_m), C_{n}, S_{n}$ (m = 1, 2) are defined in Eq. (A.2) and $f^*(q_m), C_{n}^*, S_{n}^*$ (m = 1, 2) are defined in Eq. (A.3).

Rogerson and Sandiford [18] obtained the dispersion relation in a similar form for the perfectly bonded case. Alternatively, Eqs. (3.6) and (3.7) can be also written as

$$q_1^2 S_1^2 \left[ f^*(q_1^2)^2 \Delta_1 + f^*(q_1^2) \Delta_10 + \Delta_11 \right] - q_2^2 S_2^2 \left[ f^*(q_2^2)^2 \Delta_2 + f^*(q_2^2) \Delta_20 + \Delta_21 \right]$$

$$+ \left( \frac{k_b}{k_c} \right) \left[ f^*(q_1^2) - f(q_1^2) \right] \left[ f(q_1^2) - f(q_2^2) \right] \left[ f(q_1^2) - f(q_3^2) \right] \left[ f(q_1^2) - f(q_4^2) \right] \left[ f(q_1^2) - (q_1)^2 S_1 C_2 \right] \left[ f(q_1^2) - (q_1)^2 S_1 C_2 \right]$$

$$= 0.$$
for the outer layers where
and without loss of generality the roots
are obtained for the inner layer where

\[ S_{1} \text{ and } S_{2} \] of the outer layer (inner layer) material \([10]\). In addition, spurious roots may also be found at
roots of Eq. (3.8) given by

\[ \beta_{\alpha} \]

It is noted that the expressions given in Eq. (2.12) of \([18]\) for the perfectly bonded case contain some errors.

In order to analyze the dispersion relation, the non-dimensional phase speed \(\xi = \rho v^{2}/\gamma\) is introduced so that Eq. (2.6) yields

\[ q^{12} + (\xi - 2\beta^{*}q^{2})q^{1} + (\bar{\alpha} - \xi) = 0, \quad q^{12} \]

and without loss of generality the roots \(q_{1}^{12}\) and \(q_{2}^{12}\) of Eq. (4.1) can be written as

\[ q_{1}^{12}, q_{2}^{12} = \frac{1}{2}(2\bar{\beta} - \xi \mp \sqrt{(\xi - 2\beta^{*}q^{2} - 4(\bar{\alpha} - \xi))}), \quad q_{1}, q_{2} \]

for the outer layers where \(\bar{\beta} = \beta^{*}/\gamma^{*}, \bar{\alpha} = \alpha^{*}/\gamma^{*}\). Similar equations

\[ q^{13} + (\xi^{*} - 2\beta^{*}q^{2})q^{1} + (\bar{\alpha}^{*} - \xi) = 0, \quad q^{13} \]

\[ q_{1}^{13}, q_{2}^{13} = \frac{1}{2}(2\bar{\beta}^{*} - \xi^{*} \mp \sqrt{(\xi^{*} - 2\beta^{*}q^{2} - 4(\bar{\alpha}^{*} - \xi^{*}))}), \quad q_{1}^{*}, q_{2}^{*} \]

are obtained for the inner layer where \(\bar{\beta}^{*} = \beta^{*}/\gamma^{*}, \bar{\alpha}^{*} = \alpha^{*}/\gamma^{*}, \xi^{*} = \rho^{*}v^{2}/\gamma^{*}\) and \(\rho = \rho^{*}/\rho\).

The common factor \(q_{1}^{12}q_{2}^{13}(q_{1}^{12} - q_{2}^{13})(q_{1}^{12} - q_{2}^{13})\) taken out from the denominator of Eq. (3.5) leads to spurious roots of Eq. (3.8) given by

\[ \xi = \xi_{31} = \bar{\alpha}^{*} \quad \text{(when } q_{1} = 0 \text{ or } q_{2} = 0), \quad \xi = \xi_{32} = 2(\bar{\beta} - 1 + \sqrt{\bar{\alpha} - 2\bar{\beta} + 1}) \quad \text{(when } q_{1}^{12} = q_{2}^{13}), \]

\[ \xi^{*} = \xi_{31}^{*} = \bar{\alpha}^{*} \quad \text{(when } q_{1}^{12} = 0 \text{ or } q_{2}^{13} = 0), \quad \xi^{*} = \xi_{32}^{*} = 2(\bar{\beta}^{*} - 1 + \sqrt{\bar{\alpha}^{*} - 2\bar{\beta}^{*} + 1}) \quad \text{(when } q_{1}^{12} = q_{2}^{13}), \]

where \(\xi_{31}^{*} \) and \(\xi_{32}^{*} \) are upper bounds on the phase speed of pure surface waves propagating in a half space of the outer layer (inner layer) material \([10]\). In addition, spurious roots may also be found at \(\xi = 2(\beta - 1 - \sqrt{\alpha - 2\beta + 1}) \) when \(q_{1}^{12} = q_{2}^{13}\) and \(\xi^{*} = 2(\beta^{*} - 1 - \sqrt{\alpha^{*} - 2\beta^{*} + 1}) \) when \(q_{1}^{13} = q_{2}^{12}\).

4. Analysis of dispersion relation

When \(k_{h} \rightarrow 0\) the thickness of the layers are very small compared to the wavelength. By considering small argument expansions of the hyperbolic functions the phase speeds for the imperfect interface and perfectly bonded
cases are obtained from Eqs. (3.8) and (3.11) as
\[ \xi_0 = \frac{2[D(\beta^2 + 1 - r\alpha) + r(\beta^2 + 1 - \sigma)\alpha]}{aD + r} \]  
(4.5)
and for the fully slipping case from Eq. (3.13) as
\[ \xi_{01} = 2(\beta + 1 - \sigma), \quad \xi_{02} = \frac{2(\beta^2 + 1 - r\alpha)\alpha}{a} \]  
(4.6)
The finite phase speeds given in Eq. (4.6) correspond to the fundamental mode of extensional waves of an incompressible elastic layer with free surfaces and is in agreement with Eq. (45) of Rogerson [13].

For higher modes, which have infinite phase speed \( \xi \to \infty \) when \( kh \to 0 \), Eqs. (4.2) and (4.4) yield
\[ q_1^2 = -\xi^2 + 2\beta^2 - 1 - \frac{2\beta - \bar{\alpha} - 1}{\xi} + O(\xi^{-2}), \quad q_2^2 = 1 + \frac{2\beta - \bar{\alpha} - 1}{\xi} + O(\xi^{-2}), \]
\[ q_1^2 = -a\xi^2 + 2\beta^2 - 1 - \frac{2\beta^2 - \alpha^2 - 1}{a\xi^2} + O(\xi^{-2}), \quad q_2^2 = 1 + \frac{2\beta^2 - \alpha^2 - 1}{a\xi^2} + O(\xi^{-2}). \]  
(4.7)
It is easily seen that \( q_1 \) and \( q_1^* \) are imaginary while \( q_2 \) and \( q_2^* \) are real. By substituting Eq. (4.7) into Eq. (3.8), introducing the non-dimensional frequency parameter \( \Omega = kh\sqrt{a} \) and considering the small argument expansions of the hyperbolic functions the expression to determine the cut-off frequencies \( \Omega_C > 0 \) is obtained as
\[ \sqrt{\Omega C} \sin(\Omega_C) \sin(\sqrt{\beta D}\Omega_C) - k_s [\sqrt{\beta D} \sin(\Omega_C) \sin(\sqrt{\beta D}\Omega_C) + r \sin(\Omega_C) \cos(\sqrt{\beta D}\Omega_C)] = 0. \]  
(4.8)
The cut-off frequencies thus obtained depend only on the non-dimensional parameters \( a, D, r \) and \( k_s \) and the equations for cut-off frequencies of the perfectly bonded and fully slipping cases may also be deduced from (4.8).

The cut-off frequencies for the fully slipping interface which are \( \Omega_C = n\pi, \pi/\sqrt{\beta D} \) \( (n = 1, 2, \ldots) \), correspond to the cut-off frequencies for extensional waves of an incompressible elastic layer with free surfaces (Eq. (3.8c) of [5]). It is expected that for a composite with a fully slipping interface at the low wavenumber limit the motion of the inner and outer layers are uncoupled.

4.2. High wavenumber limit, \( kh \to 0 \)

When \( kh \to \infty \) the thickness of the layers are very large compared to wavelength and the propagation behavior is similar to waves in semi-infinite media and infinite composite media. The behavior of the dispersion relation in this region depends on the roots \( q_1, q_2, q_1^* \) and \( q_2^* \) which may either real, pure imaginary or complex conjugates.

4.2.1. Roots \( q_1, q_2, q_1^* \) and \( q_2^* \) are real or complex conjugates with non-zero real part
When \( kh \to \infty \) the non-dimensional frequency \( \Omega = kh\sqrt{a} \) and \( k_s \) \( (m = 1, 2) \). For the imperfectly bonded interface dividing Eq. (3.8) by \( C_1^*C_2^*C_2^* \) and taking the limit \( kh \to \infty \), yields
\[ R(\xi)I_0(\xi) = 0, \]  
(4.9)
where \( R(\xi) = 0 \) and \( I_0(\xi) = 0 \) are the equations for the phase speeds of the Rayleigh surface wave of the outer layer \([2]\) and the Stoneley interfacial wave for fully slipping half spaces given by
\[ R(\xi) = q_1f(q_1^2)^2 - q_2g(q_2^2)^2 = 0, \]
\[ I_0(\xi) = \left[ f(q_1^2) - f(q_1^2)q_1^2f'(q_1^2)^2 + q_2^2f'(q_2^2)^2\right] + r\left[q_1f(q_1^2)^2 - q_2g(q_2^2)^2\right] = 0. \]  
(4.10)
Similarly analyzing Eqs. (3.11) and (3.13) yields
\[ R(\xi)I_0(\xi) = 0, \]  
(4.11)
for the perfectly bonded interface and Eq. (4.9) for the fully slipping interface, where \( IP(\xi) = 0 \), is the equation for the phase speed of Stoneley interfacial waves for perfectly bonded half spaces [3] given by

\[
IP(\xi) = q_1^2 q_2 [f(q_1) - f'(q_2)] - \frac{q_2^2 [f(q_1) - f'(q_2)]^2}{q_1^2} - q_1^2 q_2 [f(q_2) - f'(q_2)] - \frac{q_2^2 [f(q_2) - f'(q_2)]^2}{q_1^2} - q_1^2 q_2 [f(q_2) - f'(q_2)] - \frac{q_2^2 [f(q_2) - f'(q_2)]^2}{q_1^2} = 0. 
\] (4.12)

The roots of Eqs. (4.10) and (4.12) are denoted by \( \xi_{E}, \xi_{S}, \text{and} \xi_{P} \), respectively.

4.2.2. At least one of the roots \( q_1, q_2, q_2^* \) are pure imaginary

When \( k L \to \infty \) the hyperbolic functions \( C_{m_1}, S_{m_1}, C_{m_2}, \text{or} S_{m_2} \) (\( m = 1, 2 \)) with pure imaginary arguments are finite and the dispersion relation Eq. (3.8) yields for all values of \( \xi_{C} \) phase speeds that will tend to the limiting phase speed of the bimaterial composite \( \xi_{CL} \) [18], given by

\[
\xi_{CL} = \min \left( \xi_{E}, \frac{\xi_{S}}{a} \right), 
\] (4.13)

in which,

\[
\xi_{E} = \begin{cases} 
\xi_{E1} = \bar{a}, & \bar{a} \leq 2\hat{\beta} \\
\xi_{E2} = 2(\bar{a} - 1 + \sqrt{\bar{a}^2 - 2\hat{\beta} + 1}), & \bar{a} > 2\hat{\beta} 
\end{cases} \quad \text{(for outer layers)}
\]

\[
\xi_{S} = \begin{cases} 
\xi_{S1} = \bar{a}^*, & \bar{a}^* \leq 2\tilde{\beta} \\
\xi_{S2} = 2(\bar{a}^* - 1 + \sqrt{\bar{a}^*^2 - 2\tilde{\beta}^* + 1}), & \bar{a}^* > 2\tilde{\beta} 
\end{cases} \quad \text{(for inner layer)}
\] (4.14)

where \( \xi_{E} \) and \( \xi_{S} \) are limiting phase speeds of the outer layers and the inner layer, respectively. Hence, there are four cases of limiting phase speeds of the composite—Case 1: \( \xi_{CL} = \xi_{E1} \); Case 2: \( \xi_{CL} = \xi_{E2} \); Case 3: \( \xi_{CL} = \xi_{S1}/a \); Case 4: \( \xi_{CL} = \xi_{S2}/a \).

5. Numerical results

In the examples considered in this section incompressible isotropic materials with Mooney–Rivlin and Varga strain energy functions are considered.

5.1. Strain energy functions

The strain energy function \( W^\text{MR} \) of Mooney–Rivlin material [11] is

\[
W^\text{MR} = \frac{1}{2} \mu_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{1}{2} \mu_2 (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3), 
\] (5.1)

where \( \mu_1 \) and \( \mu_2 \) are material constants. Using Eq. (2.7) the parameters \( \bar{a} \) and \( \bar{b} \) of this material are expressed as

\[
\bar{a} = \frac{\lambda_1^2}{\lambda_2^2}, \quad 2\bar{b} = \bar{a} + 1. 
\] (5.2)

The strain energy function \( W^\text{V} \) of Varga material [11] is

\[
W^\text{V} = 2\mu (\lambda_1 + \lambda_2 + \lambda_3 - 3), 
\] (5.3)

where \( \mu \) is a material constant and from Eq. (2.7) the parameters \( \bar{a} \) and \( \bar{b} \) of this material are given by

\[
\bar{a} = \frac{\lambda_1^2}{\lambda_2^2}, \quad \bar{b} = \frac{\lambda_1}{\lambda_2}. 
\] (5.4)

The parameters \( \bar{a}^* \) and \( \bar{b}^* \) for the inner layer are similarly obtained.
5.2. Examples

The phase speed–wavenumber plots and frequency–wavenumber plots can be obtained either from Eq. (3.8) by disregarding the spurious non-dispersive roots or directly from Eq. (3.5). Here the latter method is used to obtain the dispersion curves and the examples considered correspond to the four cases referred to in Section 4.2.2. In Examples 1–4, for a given state of pre-stress the parameters $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\alpha}^\ast$ and $\tilde{\beta}^\ast$ are computed while the parameters $\sigma$, $\kappa$, $\sigma$, $\kappa$, and $D$ are prescribed. For each example the shear spring parameter is prescribed as $\kappa = 0, 1, 100$ and $\infty$ and the phase speeds of the fundamental mode $\xi^{(1)}$ and the next 15 modes $\xi^{(n+1)} (n = 1, \ldots, 15)$ are plotted for $0 \leq kh \leq 50$ while the frequencies of the fundamental mode $\Omega^{(1)}$ and the next 15 modes $\Omega^{(n+1)} (n = 1, \ldots, 15)$ are plotted for $0 \leq kh \leq 20$. The dispersion curves for Examples 1–4 show that when $\kappa = 100$ the dispersion curves are similar to the curves of the perfectly bonded case ($\kappa \to \infty$), and on the other hand when $\kappa = 1$ the dispersion curves are similar to the fully slipping case ($\kappa = 0$). The limiting phase speeds calculated from Eqs. (4.5), (4.6), (4.10) and (4.12)–(4.14) are given in Table 1.

![Figure 2](image-url)

*Fig. 2. Phase speed and frequency of the fundamental mode and next 15 modes of Example 1 ($\tilde{u} = \tilde{\beta} = \tilde{\alpha}^\ast = \tilde{\beta}^\ast = 1$; $\sigma = 0.25, a = 1.25$, $\kappa = 1.0, D = 1$): (a) and (b) phase speed; (c) and (d) frequency.*
Table 1
Limits of non-dimensional phase speed $\xi$

<table>
<thead>
<tr>
<th>$kh \to 0$</th>
<th>$kh \to \infty$</th>
<th>$\xi_0$</th>
<th>$\xi_0^1$</th>
<th>$\xi_0^2$</th>
<th>$\xi_S^1/\alpha$</th>
<th>$\xi_S^2/\alpha$</th>
<th>$\xi_L$</th>
<th>$\xi_L^*$</th>
<th>$\xi_P$</th>
<th>$\xi_IS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>4.0</td>
<td>6.0</td>
<td>3.6</td>
<td>$1^*$</td>
<td>--</td>
<td>0.8$^a$</td>
<td>--</td>
<td>0.8</td>
<td>0.568</td>
<td>0.782</td>
</tr>
<tr>
<td>Example 2</td>
<td>1.892</td>
<td>5.0</td>
<td>1.406</td>
<td>$1^*$</td>
<td>--</td>
<td>1.25$^b$</td>
<td>1.67</td>
<td>0.771</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Example 3</td>
<td>1.144</td>
<td>5.441</td>
<td>0.606</td>
<td>2.441$^*$</td>
<td>--</td>
<td>0.813$^b$</td>
<td>0.813</td>
<td>2.354</td>
<td>--</td>
<td>0.803</td>
</tr>
<tr>
<td>Example 4</td>
<td>7.85</td>
<td>7.7</td>
<td>8.0</td>
<td>--</td>
<td>4.4$^c$</td>
<td>5.0$^c$</td>
<td>4.4</td>
<td>3.937</td>
<td>--</td>
<td>4.157</td>
</tr>
</tbody>
</table>

$^a$ The limiting phase speed of outer layers is $\xi_L$.
$^b$ The limiting phase speed of inner layer is $\xi_L^*$.

Example 1. Both inner and outer layers are equi-biaxially deformed in $(x_1x_2)$-plane, i.e. $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^* = \lambda_2^* = \lambda^*$ and from Section 2 for any strain energy function $\bar{\alpha} = \bar{\beta} = \bar{\alpha}^* = \bar{\beta}^* = 1$. The other prescribed parameters are $r = 0.25$, $\alpha = 1.25$, $\sigma = -1.0$ and $D = 1$. This example corresponds to Case 3 in Section 4.2.2 with a limiting phase speed of the composite $\xi_{CL} = \xi_{S1}/\alpha = 0.8$. Phase speed and frequency plots are given in Fig. 2. When $kh \to 0$ for $k_x > 0$, $\xi^{(1)} \to \xi_0 = 4.0$ and for all other higher modes $\xi^{(n+1)} \to \infty (n = 1, 2, \ldots)$.

Fig. 3. Phase speed and frequency of the fundamental mode and next 15 modes of Example 2 ($\bar{\alpha} = \bar{\beta} = 1, \bar{\alpha}^* = 9, \bar{\beta}^* = 3, r = 1, \alpha = 6.4, \sigma = -0.5, D = 1$): (a) and (b) phase speed; (c) and (d) frequency.
while for the fully slipping case $\xi(1) \rightarrow \xi_0 = 3.6$, $\xi(2) \rightarrow \xi_0 = 6.0$ and for all other higher modes $\xi(n+2) \rightarrow \infty$ ($n = 1, 2, \ldots$). When $kh \rightarrow \infty$ for the perfectly bonded case $\xi(1) \rightarrow \xi_R = 0.568$, $\xi(2) \rightarrow \xi_{IP} = 0.782$ and for all other higher modes $\xi(n+2) \rightarrow \xi_{CL}$ ($n = 1, 2, \ldots$). From Eq. (3.8) it can be seen that when $kh/k_x \rightarrow \infty$ the phase speeds for an imperfect interface should be the same as for the fully slipping case and this is confirmed in Fig. 2(b). A horizontal ghost line appears near $\xi = \xi_S = 1.0$ for all values of $k_x$. These type of so-called ghost lines are also noted in [18] for the perfectly bonded composite. The frequency plots shown in Fig. 2(c) and (d) indicate that when $kh \rightarrow 0$ for $k_x > 0$, only the frequency of the fundamental mode $\Omega(1) \rightarrow 0$; while for $k_x = 0$ the frequencies of first two modes $\Omega(n) \rightarrow 0$ ($n = 1, 2$) which correspond to the two finite phase speeds given by Eq. (4.6). The frequencies of the higher modes tend to the cut-off frequencies obtained from Eq. (4.8).

Example 2. The outer layers are equi-biaxially deformed in $(x_1, x_2)$-plane, i.e. $\bar{\alpha} = \bar{\beta} = 1$ and the inner layer is Varga material in a state of plane strain, i.e. $\lambda_3^* = 1$ and $\lambda_1^* = \lambda_2^* = \lambda^*$ which yields $\bar{\alpha}^* = \lambda^{1/2}$ and $\bar{\beta}^* = \lambda^{1/2}$. Here $\lambda^*$ is prescribed as $\lambda^* = \sqrt{3}$ which yields $\bar{\alpha}^* = 9$, $\bar{\beta}^* = 3$ and the other prescribed parameters are $r = 1$, $a = 6.4$, $\sigma = -0.5$, and $D = 1$. This example corresponds to Case 1 in Section 4.2.2 with a limiting phase speed $\xi_{CL} = \xi_S = 1.0$. The phase speed plots shown in Fig. 3(a) and (b) indicate that when $kh \rightarrow 0$ for $k_x > 0$, $\xi(1) \rightarrow \xi_0 = 1.892$ and for

![Fig. 4. Phase speed and frequency of the fundamental mode and next 15 modes of Example 3 (\(\bar{\alpha} = 2.441, \bar{\beta} = 1.721, \bar{\alpha}^* = 25.629, \bar{\beta}^* = 5.063; r = 2.5, a = 20, \sigma = 0, D = 1\)): (a) and (b) phase speed; (c) and (d) frequency.](image-url)
all other higher modes $\xi^{(n+1)} \to \infty \ (n = 1, 2, \ldots)$, while for $k_0 = 0$, $\xi^{(1)} \to \xi_{02} = 1.406$, $\xi^{(2)} \to \xi_{01} = 5.0$ and for all other higher modes $\xi^{(n+2)} \to \infty \ (n = 1, 2, \ldots)$. In this example since there are no real values of $\xi_R$ and $\xi_P$ when $kh \to \infty$ for all values of $k_x$, $\xi^{(1)} \to \xi_R = 0.771$ and all other higher modes $\xi^{(n+1)} \to \xi_{CL} \ (n = 1, 2, \ldots)$. Ghost lines are observed for all values of $k_x$ above $\xi^* = \xi_{S2}/a = 1.25$ which is the limiting phase speed of the inner layer. The frequency plots shown in Fig. 3 (c) and (d) have features similar to the frequency plots of Example 1 as $kh \to 0$.

Example 3. The primary deformations of both inner and outer layers are plane strain deformations in the $(x_1, x_2)$-plane, i.e. $\lambda_3 = 1, \lambda_1 = \lambda_2 = 1$ and $\lambda_3^* = \lambda_3 = 1$. The outer and inner layers are Mooney–Rivlin and Varga materials, respectively, which yield $\bar{\alpha} = \lambda^2$, $\bar{\beta} = \lambda^2 + 1$, $\bar{\alpha}^* = \lambda^2$ and $\bar{\beta}^* = \lambda^2$. Here $\lambda$ and $\lambda^*$ are prescribed as $\lambda = 1.25$ and $\lambda^* = 2.25$ which will give $\bar{\alpha} = 4.41, \bar{\beta} = 1.721, \bar{\alpha}^* = 25.629$ and $\bar{\beta}^* = 5.063$ and the other prescribed parameters are $r = 2.5, a = 20, \sigma = 0$ and $D = 1$. This example corresponds to Case 4 in Section 4.2.2 with a limiting phase speed of the composite $\xi_{CL} = \xi_{S2}/a = 0.813$. Phase speed and frequency plots are given in Fig. 4. When $kh \to 0$ for $k_0 > 0$, $\xi^{(1)} \to \xi_0 = 1.144$ and $\xi^{(n+1)} \to \infty \ (n = 1, 2, \ldots)$, while for $k_0 = 0$ the phase speeds are $\xi^{(1)} \to \xi_{02} = 0.606$, $\xi^{(2)} \to \xi_{01} = 5.441$ and $\xi^{(n+2)} \to \infty \ (n = 1, 2, \ldots)$. In this example since $\xi_R = 2.354 > \xi_{CL}$, when $kh \to \infty$ for the perfectly bonded case $\xi^{(n)} \to \xi_{CL} \ (n = 1, 2, \ldots),$
while for the fully slipping case, $\bar{\xi}^{(1)} \rightarrow \bar{\xi}_{cL} = 0.803$ and $\bar{\xi}^{(n+2)} \rightarrow \bar{\xi}_{cL}$ $(n = 1, 2, \ldots)$. Fig. 4(b) shows that when $kh/k_x \rightarrow \infty$ the phase speeds for an imperfect interface will be the same as for the fully slipping case. Ghost lines for all values of $k_x$ are observed above $\xi = \bar{\xi}_{cL}$ which is the limiting phase speed of the composite. The frequency plots are shown in Fig. 4(c) and (d).

Example 4. Both inner and outer layers are Varga materials, the outer layers are pre-stressed by uniaxial tension in $x_1$ direction, i.e. $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{1/2}$ while the inner layer is in a state of plane strain in $(x_1, x_2)$-plane, i.e. $\lambda_3 = 1$ and $\lambda_4 = \lambda^{-3/2}$ which yield $\bar{\alpha} = \lambda^3, \bar{\beta} = \lambda^{1/2}, \bar{\gamma} = \lambda^{1/4}$ and $\bar{\beta}^* = \lambda^{1/2}$. Here $\lambda$ and $\lambda^*$ are prescribed as $\lambda = \sqrt{x/\pi}$ and $\lambda^* = 1.5$ which results in $\bar{\alpha} = 4.41, \bar{\beta} = 2.1, \bar{\gamma} = 5.063$ and $\bar{\beta}^* = 2.25$ and the other prescribed parameters are $r = 1, a = 1, \sigma = -0.75$, and $D = 1$. This example corresponds to Case 2 in Section 4.2.2 with a limiting phase speed of the composite $\bar{\xi}_{cL} = \bar{\xi}_{c2} = 4.4$. The phase speed plots shown in Fig. 5(a) and (b) show that when $kh \rightarrow 0$ for $k_x > 0$, $\bar{\xi}^{(1)} \rightarrow \bar{\xi}_0 = 7.85$ and $\bar{\xi}^{(n+1)} \rightarrow \infty (n = 1, 2, \ldots)$, while for $k_x = 0$ the phase speeds are $\bar{\xi}^{(1)} \rightarrow \bar{\xi}_0 = 7.7, \bar{\xi}^{(2)} \rightarrow \bar{\xi}_0 = 8.0$ and $\bar{\xi}^{(n+2)} \rightarrow \infty (n = 1, 2, \ldots)$. When $kh \rightarrow \infty$ for the perfectly bonded case, $\bar{\xi}^{(1)} \rightarrow \bar{\xi}_R = 3.937$ and $\bar{\xi}^{(n+1)} \rightarrow \bar{\xi}_{cL} (n = 1, 2, \ldots)$ while for the fully slipping case, $\bar{\xi}^{(1)} \rightarrow \bar{\xi}_R, \bar{\xi}^{(2)} \rightarrow \bar{\xi}_R = 4.157$ and $\bar{\xi}^{(n+2)} \rightarrow \bar{\xi}_{cL} (n = 1, 2, \ldots)$. Fig. 5(b) shows that when $kh/k_x \rightarrow \infty$ the phase speeds for an imperfect interface should be the same as for the fully slipping case. Ghost lines are observed for all values of $k_x$ above $\xi = \bar{\xi}^{(n+1)}_{cL}/(n = 5.0)$ where $\bar{\xi}^{(n+1)}_{cL}$ is the limiting phase speed of the inner layer. Frequency plots are shown in Fig. 5(c) and (d) and in this example since $\sqrt{\pi D} = 1$, for the fully slipping interface case the cut-off frequencies will coincide (see Section 4.1) and pairs of modes will have the same cut-off frequency as shown in Fig. 5(d).

6. Summary and conclusions

In the present analysis, the dispersion relation for time-harmonic extensional waves in a pre-stressed imperfectly bonded incompressible symmetric layered composite is investigated. The limiting phase speeds in the low and high wavenumber region and an expression to determine the cut-off frequencies are analytically obtained from the dispersion equation.

At high wavenumber limit $kh \rightarrow \infty$, for the perfectly bonded case, both $\bar{\xi}_L$ and $\bar{\xi}_D$ are lower than $\bar{\xi}_{cL}$, while for the fully slipping case, $\bar{\xi}^{(1)} \rightarrow \min(\bar{\xi}_L, \bar{\xi}_D)$ while $\bar{\xi}^{(2)} \rightarrow \max(\bar{\xi}_L, \bar{\xi}_D)$ and $\bar{\xi}^{(n+2)} \rightarrow \bar{\xi}_{cL}$(n = 1, 2, 3, 4). If either $\bar{\xi}_L$ or $\bar{\xi}_D$ is lower than $\bar{\xi}_{cL}$ then the speed of the fundamental mode will tend to that lower value and $\bar{\xi}^{(n+1)} \rightarrow \bar{\xi}_{cL}$(n = 1, 2, 3, 4). If neither of $\bar{\xi}_L$ and $\bar{\xi}_D$ is lower than $\bar{\xi}_{cL}$, then all branches will tend to $\bar{\xi}_{cL}$. Similar behavior will occur for the fully slipping case but the parameters are $\bar{\xi}_L$ and $\bar{\xi}_D$ instead of $\bar{\xi}_L$ and $\bar{\xi}_D$. For the imperfectly bonded case the phase speeds are similar to the speeds of the fully slipping case only when $kh/k_x \rightarrow \infty$. Ghost lines are observed in the dispersion curves and the appearance of these lines are a feature for any value of $k_x$. The numerical results indicate that these ghost lines appear above $\xi = \bar{\xi}_{cL}$.

Acknowledgements

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Appendix A. The components of propagator matrix

The components of propagator matrix $P(q)$ are given by

$$
P_{11} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{12} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{13} = q_1 f(q_1)c_1 - q_1 c_1)k^{-1},
$$
$$
P_{14} = f(q_1)c_1 - c_1)k^{-1},
$$
$$
P_{21} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{22} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{23} = q_1 f(q_1)c_1 - q_1 c_1)k^{-1},
$$
$$
P_{24} = f(q_1)c_1 - c_1)k^{-1},
$$
$$
P_{31} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{32} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{33} = q_1 f(q_1)c_1 - q_1 c_1)k^{-1},
$$
$$
P_{34} = f(q_1)c_1 - c_1)k^{-1},
$$
$$
P_{41} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{42} = y(q_1 f(q_1)c_1 - f(q_1)c_1)k^{-1},
$$
$$
P_{43} = q_1 f(q_1)c_1 - q_1 c_1)k^{-1},
$$
$$
P_{44} = f(q_1)c_1 - c_1)k^{-1},
$$

where

$$
f(q_1) = 1 + q_1^2 - \sigma,
$$
$$
C_{m} = \cosh(q_mkh),
$$
$$
S_{m} = \sinh(q_mkh),
$$
$$
\sigma = \frac{y}{\gamma},
$$
$$
\kappa = q(q_1)(\frac{q_m}{q_1} - \frac{q_2}{q_1}).
$$

(A.2)

The components of propagator matrix $P'(q)$ are obtained from the above equation by interchanging $P_{ij} \leftrightarrow P_{ji}$.

$$
q_m = q_m', f(q_m) = f(q_m'), C_m = C_m', S_m = S_m', \gamma \leftrightarrow \gamma' \text{ and } \kappa \leftrightarrow \kappa'.
$$

where

$$
f'(q_m') = 1 + q_1^2 - \sigma,
$$
$$
C_{m}' = \cosh(q_m'kh),
$$
$$
S_{m}' = \sinh(q_m'kh),
$$
$$
\gamma = \frac{\gamma}{\gamma'},
$$
$$
\kappa = \frac{\kappa}{\kappa'}
$$

(A.3)

References